Opportunistic Space-Time Block Codes

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Abstract—Rate and diversity impose a fundamental tradeoff in space-time coding. High-rate space-time codes come at a cost of lower diversity, and high reliability (diversity) implies a lower rate. In [2], [3], we proposed a different point of view where we designed high-rate space-time codes that have a high-diversity code embedded within them. This allows for a form of wireless communications where the high-rate code opportunistically takes advantage of good channel realizations while the embedded high-diversity code ensures that at least part of the information is decoded reliably. In [2], [3], we presented the code design strategy, criteria, and several constructions. In this paper, our focus is on decoding algorithms, quantifying the performance of the code examples in [2], [3] and presenting new examples.

I. INTRODUCTION

There is a fundamental tradeoff between rate and reliability (diversity) in multiple-antenna wireless communications both in multiplexing rate [8] as well as in fixed rate codes [7]. In [2], [3], we argued that diversity can be viewed as a systems resource that can be allocated judiciously to achieve a desirable rate-diversity tradeoff in multimedia wireless communications. Multimedia data is often complex and consists of various components with different rate/reliability requirements. For example, a real-time data stream is a candidate to receive more diversity (protection) than non-real-time data. Other examples include complex data streams such as compressed video, whose various components require different levels of error protection. Over-provisioning of diversity to one data component will mean the loss of rate to that and other data components, a waste of system resources. We have designed high-rate space-time codes in [2], [3] that have a high-diversity code embedded within them. This allows a form of wireless communications where the high-rate code opportunistically takes advantage of good channel realizations while the embedded high-diversity code ensures that at least part of the information is decoded reliably. The proposed opportunistic codes do not require channel knowledge at the transmitter and outperform time-sharing schemes. Furthermore, in [2], [3], we developed code design strategies and criteria and presented examples of specific code constructions for 4 transmit antennas. The focus of this paper is on the performance of the code examples in [2], [3] and comparing them with maximum-diversity codes. In addition, we present a new code construction for 3 transmit antennas with two and three layers of diversity. We also explore the design and the performance-complexity tradeoff issues for the decoding algorithms of our codes.

II. BACKGROUND

A. Transmission model

We consider the quasi-static flat–fading channel where coded information is transmitted over \( M_t \) antennas and received using \( M_r \) antennas. It is assumed that the receiver has perfect channel knowledge. The channel is constant over a coherence interval of \( T \) symbols and changes independently from one coherence interval to the next. After demodulation and sampling, the received signal can be written as

\[ Y = HX + Z, \]

where \( Y \in \mathbb{C}^{M_r \times T} \) is the received sequence, \( H \in \mathbb{C}^{M_r \times M_t} \) is the quasi-static channel fading matrix, \( X \in \mathbb{C}^{M_t \times T} \) is the space-time code matrix with transmit power constraint \( P_t \), and \( Z \in \mathbb{C}^{M_r \times T} \) is assumed to be additive white Gaussian noise with variance \( \sigma^2 \).

B. Design criteria for opportunistic codes

The main idea in [2], [3] is to design a codebook which provides different diversity and rate levels to the different information streams. Let \( A \) denote the message set from the first information stream and \( B \) denote that from the second information stream. The rates for the two message sets are, respectively, \( R(A) \) and \( R(B) \). The decoder jointly decodes the two message sets with average error probabilities, \( P_e(A) \) and \( P_e(B) \), respectively. We design the code \( X(a, b) \), such that a certain tuple \((R_a, D_a, R_b, D_b)\) of rates and diversities are achievable, where \( R_a = R(A) = \frac{\log(\bar{D}_a)}{T} \) and analogous to [8] we define

\[ D_a = \lim_{SNR \to \infty} \frac{\log P_e(A)}{\log(SNR)}, \quad D_b = \lim_{SNR \to \infty} \frac{\log P_e(B)}{\log(SNR)}. \]

For fixed rate codes it has been shown in [2], [3] that to guarantee the diversity orders \( D_a, D_b \) we need to design codes such that,

\[ \min_{a_1 \in A} \min_{b_1, b_2 \in B} \text{rank}(B(x_{a_1, b_1}, x_{a_1, b_2})) \geq D_a/M_r \]

\[ \min_{b_1, b_2 \in B} \min_{a_1, a_2 \in A} \text{rank}(B(x_{a_1, b_1}, x_{a_2, b_2})) \geq D_b/M_r. \]

where \( B \) is the codeword difference matrix. Basically, this implies that if we transmit a particular message \( a \in A \), regardless of which message is chosen in message set \( B \), we are ensured a diversity level of \( D_a \) for this message set. A similar argument holds for message set \( B \).
C. Code Examples

We present specific opportunistic code constructions given in [2], [3] for 4 transmit antennas.

Example 1: Here $A$ comes from the message set $\{a(0), a(1), a(2)\} \in S$ and $B$ comes from $b(0) \in S$. This code example achieves the tuple, $(\frac{1}{2} \log |S|, 4M_r, \frac{1}{2} \log |S|, M_r)$.

$$X_{a,b} = X_a + X_b = \begin{bmatrix} a(0) & a(1) & a(2) & b(0) \\ -a^*(1) & a^*(0) & 0 & a(2) \\ -a^*(2) & 0 & a^*(0) & -a(1) \\ 0 & -a^*(2) & a^*(1) & a(0) \end{bmatrix}$$

Example 2: Here $A$ comes from the message set $\{a(0), a(1)\} \in S$ and $B$ comes from $\{b(0), b(1)\} \in S$. This example achieves the tuple, $(\frac{1}{2} \log |S|, 3M_r, \log |S|, 2M_r)$.

$$X_{a,b} = X_a + X_b = \begin{bmatrix} a(0) & a(1) & 0 & 0 \\ -a^*(1) & a^*(0) & 0 & 0 \\ b(0) & b(1) & a^*(0) & -a(1) \\ b(2) & b(3) & a^*(1) & a(0) \end{bmatrix}$$

Example 3: Here $A$ comes from the message set $\{a(0), a(1)\} \in S$ and $B$ comes from $\{b(0), b(1), b(2), b(3)\} \in S$. This example achieves the tuple, $(\frac{1}{2} \log |S|, 3M_r, \log |S|, 2M_r)$.

$$X_{a,b} = X_a + X_b = \begin{bmatrix} a(0) & a(1) & b(2) & b(3) \\ -a^*(1) & a^*(0) & b^*(3) & -b^*(2) \\ b(0) & b(1) & a(0) & -a(1) \\ -b^*(1) & b^*(0) & a^*(1) & a^*(0) \end{bmatrix}$$

The complex equivalent channel matrix for this example:

$$H = \begin{bmatrix} h(1) & h(2) & 0 & 0 & h(3) & h(4) \\ h^*(2) & -h^*(1) & h(2) & 0 & -h^*(4) & h^*(3) \\ h^*(3) & h^*(2) & h(1) & 0 & 0 & 0 \end{bmatrix}$$

Example 4: In this example, $M_r = 4$ and the block size is $T = 5$ symbols, giving rise to a non-square code matrix design. Here $A$ comes from the message set $\{a(0), a(1)\} \in S$ and $B$ comes from $\{b(0), b(1), b(2), b(3)\} \in S$. This example achieves the tuple, $(\frac{1}{2} \log |S|, 3M_r, \frac{1}{2} \log |S|, 2M_r)$.

$$X_{a,b} = X_a + X_b = \begin{bmatrix} a(0) & -a^*(1) & b(0) & b(2) & -b^*(3) \\ a(1) & a^*(0) & b(1) & b(3) & b^*(2) \\ 0 & 0 & a^*(0) & a^*(1) & -b^*(1) \\ 0 & 0 & -a(1) & a(0) & b^*(0) \end{bmatrix}$$

In all the examples above, the rate tuples hold regardless of the common transmit alphabet $S_T = S$. Therefore, the transmit constellation as well as the constellation of the information symbols coincide. Hence the rate-diversity trade-off for alphabet constrained applies directly to these examples. Also note, that a trivial outer-bound for the rate tuple is given from the single-layer bound given in [7]. Hence we have that,

$$D_a \leq |M_T - R_a/\log(|S_T|) + 1|, \quad D_b \leq |M_T - R_b/\log(|S_T|) + 1|$$

$$\min(D_a, D_b) \leq |M_T - (R_a + R_b)/\log(|S_T|) + 1|$$

(5)

III. Three Layer Code

The next example is a new code construction for 3 transmit antennas. This code can achieve up to three different layers of diversity with rate more than one.

Example 5: In this example $M_r = 3$ and the block size is $T = 4$ symbol periods. Here $A$ comes from the message set $\{a(0), a(1), a(2)\} \in S$, $B$ comes from $b(0) \in S$ and $C$ comes from $c(0) \in S$. Therefore, this code achieves a total rate of $R = R_a + R_b + R_c = \frac{1}{2} \log |S|$.

$$X_{a,b,c} = X_a + X_b + X_c = \begin{bmatrix} a(0) & a^*(0) & c^*(0) \\ a(1) & a^*(0) & \frac{c^*(0)}{k} & -a^*(2) \\ a(2) & b^*(0) & a^*(0) & a^*(1) \end{bmatrix}$$

This code can be shown to achieve the tuple $(\frac{1}{2} \log |S|, 3M_r, \frac{1}{2} \log |S|, 2M_r, \frac{1}{2} \log |S|, 1M_r)$ with a constraint on the scaling factor $k$ which is selected to ensure diversity $3M_r$ for layer $A$ (see the Appendix for a proof).

From this example, we can also derive a higher rate $R = \frac{5}{4}$ code with only two layers of diversity $3M_r$ and $M_r$ for message sets $A$ and $C$, respectively. The achievable tuple for this code is $(\frac{1}{2} \log |S|, 3M_r, \frac{1}{2} \log |S|, 1M_r)$.

$$X_{a,c} = X_a + X_c = \begin{bmatrix} a(0) & -a^*(1) & c^*(0) \\ a(1) & a^*(0) & \frac{c^*(0)}{k} & -a^*(2) \\ a(2) & \frac{c^*(2)}{k} & a^*(0) & a^*(1) \end{bmatrix}$$

Note that in this example, the transmit alphabet $S_T \neq S$. In fact $|S_T| = 2|S|$, and hence the rate-tuple points in comparison with the single-layer rate-diversity trade-off needs to take this into account as done in (5).

IV. Receiver Architectures

The additive codeword examples, due to their linearity over the complex field, are amenable to computationally-efficient lattice decoding strategies, such as the sphere decoder [4]. Alternatively, decoding can also be performed using successive interference cancellation (IC) where the more reliable high-diversity layer is decoded first and its effect on the received signal is cancelled followed by decoding of the lower-diversity layer. Successive IC achieves near-ML performance for the low-diversity layer while it suffers some degradation for the high-diversity layer (due to error propagation). IC performance can be improved by using a hybrid ML/IC algorithm which selects one message set on which a full-length ML search is performed, while successive IC is applied on the remaining message set(s).

As an example, we describe the hybrid ML/IC receiver for code Example 3 next. For all possible $\{a(0), a(1)\} \in A$, the effect of message set $A$ is cancelled from the received signal followed by decoding of $\{b(0), b(1), b(2), b(3)\} \in B$ which requires only a simple matched filter operation. The hybrid ML/IC algorithm reduces the decoding complexity of Example 3 from a full ML search of size $4^3 = 4096$ to a size $4^2 = 16$ search plus a simple matched filter operation for each of the 16 choices of $\{a(0), a(1)\}$.
V. NUMERICAL RESULTS

Unless otherwise stated, we assume QPSK modulation, a Rayleigh flat-fading channel, and joint ML decoding. Figures 1-4 depict the performance of Examples 1-4 with joint ML decoding and ideal CSI. In Figure 1, we show the performance with successive IC decoding and illustrate how the performance degradation of the high-diversity layer in IC decoding can be eliminated by using the hybrid ML/IC algorithm. In Figure 3, we investigate the resulting performance degradation when the assumption of perfect CSI at the receiver is not satisfied. The coherent ML decoder uses estimated CSI acquired by transmitting a rate-$\frac{3}{4}$ orthogonal pilot codeword (Octonion structure)[5] and using simple matched filtering at the receiver to calculate the CSI. We observe a 3 dB SNR penalty at high SNR due to channel estimation error.

In Figures 5-7, we compare code examples 1-4 with the Octonion code [5] using the measure of Effective Throughput $\eta$ defined as $\eta = (1 - FER_A) \cdot R_A + \log_2(M) + (1 - FER_B) \cdot R_B + \log_2(M)$, where $M$ is the constellation size, $FER_A$ and $FER_B$ denote the frame error rates in message sets $\mathcal{A}$ and $\mathcal{B}$, respectively. We assume a total power constraint $P$ for each codeword, QPSK modulation and 20 codewords per frame. The figures show that at high SNR (where $FER \approx 0$), our opportunistic codes achieve a higher throughput level of 2 bits (for Example 1), 3 bits (for Example 2,3) and 2.4 bits (for Example 4) per channel use (PCU) whereas the achievable throughput for the Octonion is 1.5 bits PCU. We can also observe different cross-over points at input SNR levels of 14 dB, 24 dB, 15.5 dB and 16 dB for Examples 1, 2, 3 and 4, respectively. Note that if we know the operating SNR regime, we can choose the opportunistic code so as to dominate the single-layer code.

Finally, figure 8 depicts the performance of the new code Example 5 with joint ML decoding under ideal and estimated CSI for $k = 1.6$. Figure 9 compares example 5 with 3 TX orthogonal rate-$\frac{3}{4}$ code [6] in terms of effective throughput. At high SNR our proposed code achieves an effective throughput level of 2.5 bits PCU, whereas the achievable throughput for the 3 TX orthogonal code is 1.5 bits PCU. The cross-over point is at an input SNR level of 14 dB. Figure 10 shows that the message set with diversity-3 layer of this code at rate-$\frac{3}{4}$ with QPSK modulation achieves lower BER than the 3 TX orthogonal rate-$\frac{3}{4}$ code in [6] with 16QAM modulation and the 3 TX full-rate non-orthogonal code in [9] with 8PSK modulation, all at the same transmission spectral efficiency of 3 bits PCU.

APPENDIX

Diversity Proof of Example 5:

We need to ensure that for $a_1 \neq a_2$, $(X_{a_1, b_1, c_1} - X_{a_2, b_2, c_2})$ is full-rank for any $\{b_1(0), b_2(0)\} \in \mathcal{B}$ and...
Octonion Code
Perfect and Estimated CSI for $k$ and the Octonion Code.

Fig. 7. Throughput Comparison between Example 4 and the Octonion Code.

Fig. 8. Performance of Example 5 (Rate-$\frac{5}{4}$) with ML Decoding under Perfect and Estimated CSI for $k = 1.6$.  

Fig. 9. Throughput Comparison between Example 5 and the 3 TX Rate-$\frac{1}{4}$ Orthogonal Code.

Fig. 10. Performance Comparison of Example 5 (Rate-$\frac{5}{4}$) with Orthogonal Rate-$\frac{3}{4}$ and Non-orthogonal Full-Rate Codes at 3 bits PCU.

\[
\{c_1(0), c_2(0)\} \in \mathcal{C}, \text{ i.e., } a_1 \min_{a_2 \in A} \min_{b_1, b_2, c_1, c_2 \in C} \text{rank}(X_{a_1, b_1, c_1} - X_{a_2, b_2, c_2}) = 3M_r
\]

Suppose there exists $\tilde{a} = a_1 - a_2 \neq 0$, $\tilde{b} = b_1 - b_2$, and $\tilde{c} = c_1 - c_2$ such that rank$(X_{\tilde{a}, \tilde{b}, \tilde{c}}) = 1$, which implies that there exist $\alpha, \beta$ such that both of the following hold,

\[
\begin{bmatrix}
\tilde{a}(1) \\
\tilde{a}^*(1) \\
\tilde{a}^*(2)
\end{bmatrix} = \alpha \begin{bmatrix}
\tilde{a}(0) \\
\tilde{a}^*(0) \\
\tilde{a}^*(2)
\end{bmatrix}
\]

(6)

\[
\begin{bmatrix}
\tilde{a}(1) \\
\tilde{b}^*(0) \\
-\tilde{a}^*(2)
\end{bmatrix} = \beta \begin{bmatrix}
\tilde{a}(2) \\
\tilde{b}^*(0) \\
\tilde{a}^*(0)
\end{bmatrix}
\]

(7)

From (6) we can write

\[
\tilde{a}(0) = \alpha \tilde{a}(1) \Rightarrow \tilde{a}^*(0) = \alpha^* \tilde{a}^*(1)
\]

(8)

\[
-\tilde{a}^*(1) = \alpha \tilde{a}^*(0) \Rightarrow \tilde{a}^*(0) = -\frac{1}{\alpha} \tilde{a}^*(1)
\]

If $\tilde{a}(0) \neq 0$ and (or) $\tilde{a}(1) \neq 0$, the relationship in (8) yields $\alpha^* = -\frac{1}{\alpha}$ which is not possible. However, when both $\tilde{a}(0) = 0$ and $\tilde{a}(1) = 0$, the relationship in (7) results in $\tilde{a}(2) = 0$, and hence $\tilde{a} = \{\tilde{a}(0), \tilde{a}(1), \tilde{a}(2)\} = 0$ yielding...
a contradiction. Hence we have $D_a \geq 2M_r$ and we will show shortly in the following paragraphs that $D_a = 3M_r$. Again, when $a(0) = 0 = \tilde{a}(1)$, (6) further yields
\[
-\tilde{a}^*(2) = \frac{\tilde{b}^*(0) - \tilde{a}^*(1)}{k} \Rightarrow \alpha = \frac{-\tilde{a}^*(2)k}{\tilde{b}^*(0)};
\]
\[
\tilde{c}^*(0) = -\alpha \tilde{a}^*(2) \Rightarrow -\tilde{a}^*(2) = \frac{1}{\alpha} \tilde{c}^*(0) \tag{9}
\]
From (9) it is easy to show that for $k^2 \neq \frac{\tilde{b}^*(0)}{(\tilde{a}^*(2))^2}$ the relationship in (6) always results in contradiction and hence does not hold. Likewise, we can show by a similar argument that (7) also results in a contradiction by properly choosing $k$ as a function of the $MPSK$ constellation size used. Therefore, we chose $k$ to have a contradiction in both (6) and (7). For example we have shown that, for QPSK, $k$ can be any positive real number not equal to $1, \sqrt{2}$ and $\frac{1}{\sqrt{2}}$. Similarly, we can easily enumerate the permissible values of $k$ for higher-order $MPSK$ constellations.

Let $b(0) \neq 0$, and suppose $D_a = 1$. Therefore, again as before, there exists $\alpha \neq 0, \beta \neq 0$ such that (6)-(7) hold. From the relationship in (7) we can write $\tilde{a}(1) = \beta \tilde{a}(2)$, $\tilde{a}^*(2) = -\beta \tilde{a}^*(1)$ which again yields $|\beta|^2 = -1$ if $\tilde{a}(1) \neq 0$ and (or) $\tilde{a}(2) \neq 0$. Therefore, we necessarily need $\tilde{a}(1) = \tilde{a}(2)$, which from (6) implies that $b(0) = 0$ yielding a contradiction. Hence $D_a \geq 2M_r$. However, $D_b \neq 3M_r$, as $b(0)$ in the given codeword is transmitted using only two antennas, hence $D_b = 2M_r$. The diversity for message set $C$, $D_b = M_r$ is apparent from the code design.

We will next show that the diversity for message set $A$ cannot be $2M_r$ as well. For notational brevity, let $X, Y, Z$ represent the first, second and third columns of the $X_{a,b,c}$. Suppose there exists $\tilde{a} \neq 0$ and $b, c$ such that $rank(X_{a,b,c}) = 2$. This implies that for such a choice, a column of $X_{a,b,c}$ can be expressed as a linear combination of the other two columns. Let us assume that $X = \alpha Y + \gamma Z$. From (6)-(7) we have already shown that $X \neq \alpha Y \Rightarrow X - \alpha Y = \epsilon_1$ and $X \neq \gamma Z \Rightarrow Z = X - \epsilon_2$. Combining these 2 relations, we get
\[
2X - \alpha Y - \gamma Z = \epsilon_1 - \epsilon_2
\]
\[
\Rightarrow X = \frac{\epsilon_1 - \epsilon_2}{2} - \frac{\alpha}{2}Y + \frac{\gamma}{2}Z
\]
Since this relationship holds for any $\alpha$ and $\gamma$, if we choose in particular $\alpha = 2a^*$ and $\gamma = 2b^*$, we can write $X = \alpha^*Y + \gamma^*Z = \frac{\epsilon_1 - \epsilon_2}{2}$. Now we need to show that $\frac{\epsilon_1 - \epsilon_2}{2} \neq 0$, and hence, $X \neq \alpha^*Y + \gamma^*Z$.

**Case 1:** $\frac{\epsilon_1 - \epsilon_2}{2} = 0$ when both $\epsilon_1 = 0$ and $\epsilon_2 = 0$. This is not possible since it leads to $rank(X_{a,b,c}) = 1$.

**Case 2:** $\frac{\epsilon_1 - \epsilon_2}{2} = 0$ when $\epsilon_1 = \epsilon_2$.

Suppose $X_{a,b,c}$ and $X_{b,a,c}$ are two distinct codewords. The codeword difference matrix is given by
\[
D = X_{a,b,c} - X_{a,b,c}
\]
\[
= \begin{bmatrix}
\tilde{a}(0) - \tilde{a}(1) & \tilde{a}(1) - \tilde{a}(2) & \tilde{a}(2) \\
-\tilde{a}^*(1) & -\tilde{a}^*(0) + \tilde{b}^*(0) \\
-\tilde{a}^*(2) & \tilde{b}^*(0) - \tilde{a}^*(1) \\
\tilde{c}^*(0) & \tilde{c}^*(1) - \tilde{a}^*(2)
\end{bmatrix}
\tag{10}
\]
Therefore,
\[
\epsilon_1 = \begin{bmatrix}
\tilde{a}(0) - \alpha \tilde{a}(1) \\
-\tilde{a}^*(1) - \alpha \tilde{a}^*(0) \\
-\tilde{a}^*(2) - \alpha \tilde{a}^*(1)
\end{bmatrix} ; \quad \epsilon_2 = \begin{bmatrix}
\tilde{a}(0) - \gamma \tilde{a}(2) \\
-\tilde{a}^*(2) - \gamma \tilde{a}^*(0) \\
\tilde{c}^*(0) - \gamma \tilde{a}^*(1)
\end{bmatrix}
\]

We will show by contradiction that $\epsilon_1$ and $\epsilon_2$ cannot be equal for any choice of $\tilde{a}, b, c$ and for any particular values of $\alpha$ and $\gamma$. Let us assume that $\epsilon_1 = \epsilon_2$.
\[
\epsilon_1(2) = \epsilon_2(2) \Rightarrow -\tilde{a}^*(1) - \alpha \tilde{a}^*(0) = -\tilde{a}^*(2) - \gamma \tilde{b}^*(0)
\]
\[
\Rightarrow \alpha = \gamma \tilde{b}^*(0)
\tag{11}
\]
Replacing $\alpha$ in $\epsilon_1(3)$ by (11) results in a contradiction on the value of $k$ as follows
\[
\epsilon(3) = \epsilon(3) \Rightarrow -\tilde{a}^*(2) - \alpha \tilde{b}^*(0) = -\tilde{a}^*(2) - \gamma \tilde{b}^*(0)
\]
\[
\Rightarrow -\tilde{b}^*(0) \tilde{b}^*(0) = -\tilde{b}^*(0) \tilde{b}^*(0)
\]
\[
\Rightarrow k^2 = \frac{\tilde{b}^*(0) \tilde{b}^*(0)}{\tilde{b}^*(0) \tilde{b}^*(0)}\tag{12}
\]
Equation (12) exhibits a clear contradiction according to the prior condition set on $k$. Therefore we can claim that $\epsilon_1 \neq \epsilon_2 \Rightarrow \frac{\epsilon_1}{k^2} \neq 0$ and $X \neq \alpha^*Y + \gamma^*Z$. Similarly, we can show that $Y$ (and $Z$) cannot be expressed as any linear combination of the remaining two columns. This completes the proof that for $\tilde{a} \neq 0$ and for any $\tilde{b}, c$ the rank of the matrix $X_{a,b,c}$ is 3 and the message set $A$ always achieves diversity $D_a = 3M_r$.

**References**


