On the Achievable Rates of Time-Varying Frequency-Selective Channels

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Abstract

The mutual information of white Gaussian codebooks over time-varying frequency-selective channels is investigated. Unlike most previous work, the channel is assumed to vary within the transmission block. The effects of tap correlation in time and between channel taps at a given time on mutual information are studied. This study leads to observations posed as open questions.

I Introduction

One distinguishing characteristic of wireless communication is that the transmission environment could vary rapidly in time. This effect, commonly termed fading, occurs due to mobility and the variation depends on both the transmission frequency, bandwidth as well as the amount of mobility. In addition, for broadband transmission, the multipath delay-spread in a scattering environment translates to inter-symbol interference (ISI) in digital transmission. In this paper we are interested in investigating the effects of fast time-variation of the channel, when the channel varies within a transmission block in broadband wireless environment.

Information-theoretic aspects of point-to-point wireless transmission have been a subject of significant research in the recent past (see [1] for an overview). Most of the research has focussed on either the narrowband case where channel is time-varying but is not frequency-selective, or the slowly fading assumption is used in broadband transmission, i.e. the channel remains constant during a transmission block (block time-invariance), and varies across transmission blocks. This decoupling of the frequency-selectivity (ISI) and the time-variation (fading) is usually justified by arguing that the transmission bandwidth is much larger than the Doppler spread [1]. In calculating the mutual information using this assumption, the transmission block length becomes large and as done in [2], the achievable rate $I$ is written as

$$I = (2\pi)^{-1} \mathbb{E} \left[ \int_{0}^{2\pi} \log(1 + \frac{|H(f)|^2 S(f)}{N(f)}) df \right],$$  \hspace{1cm} (1)

where the expectation is taken over the random channel frequency response $H(f)$ and the signal, noise spectra are $S(f), N(f)$ respectively. This expression is derived [2] asymptotic in both the transmission block length and the number of transmission blocks over which the coding is done. In time-varying channels there is an inherent conflict between increasing the transmission block length (for coding arguments) and the block time-invariance assumption. Therefore, in the literature where the block time-invariance assumption is made, the effects of Doppler spread, i.e. how fast the channel varies in time, is washed away. This block time-invariance assumption is also violated when there are synchronization errors such as a frequency offset, which manifests itself as a fast time-varying multiplicative noise. Therefore, in this paper we do not make the block time-invariance assumption and investigate the achievable rates for time-varying ISI channels. This allows us to examine the effect of channel time-correlation (or Doppler spread) on achievable rates. In this paper we make two main assumptions. One is that the channel is perfectly tracked in the receiver (but the transmitter does not have channel state information) and the other is that we restrict our attention to white Gaussian input codebooks. The first assumption could be justified even in fast time-varying channels by using conditional channel estimates (such as done in per-survivor processing [3, 4]). However, our goal is to isolate and understand the effects due to time-variation on mutual information using this assumption. Perhaps the effects of channel estimation errors can also be studied using techniques similar to [5]. The second assumption is more for convenience, and other input spectra and transmission bases functions could also be evaluated using the techniques developed. The main contributions of this paper are

- Compact recursive representation of mutual information in terms of Cholesky factors.
- Numerical investigation of the effects of time-correlation and tap-correlation on mutual information of time-varying ISI channels.

The paper is organized as follows. In Section II we outline our notation and describe the channel model. Section III contains the main calculation of the paper where the recursive formula for mutual information is
developed. In Section IV we investigate special cases of this result and the effect of time/tap correlations on mutual information. Finally a numerical investigation is done in Section V.

II Channel Model and Assumptions

We consider a time–varying frequency–selective channel and model its impulse response at time $k$ as a finite impulse response (FIR) filter with memory $\nu$ denoted by $h^{(k)} = [h_1^{(k)} \cdots h_{\nu+1}^{(k)}]$. Therefore, the sampled received signal $y(k)$ is given by

$$y(k) = \sum_{n=1}^{\nu+1} h_n^{(k)} x(k-n+1) + z(k),$$  \hspace{1cm} (2)

where $\{x(k)\}$ is the transmitted codeword and $z(k)$ is the additive white Gaussian noise. For illustration, we assume that $\nu = 1$, i.e., $h^{(k)} = [h_1^{(k)} \, h_2^{(k)}]$. The two channel taps are assumed to be stationary ergodic Gaussian random processes with zero mean and variance equal to 0.5. Note that the block time-invariant model does not produce stationary channel taps. A model for the time variation of each tap is discussed in Section IV-C. At first, we assume the 2 taps to be uncorrelated then we study the effect of tap correlation in Section IV-D.

The input and noise processes are assumed zero–mean white Gaussian and we denote the ratio of their variances by SNR.

We consider transmission in blocks of length $N$, with the transmission blocks isolated from one another using guard sequences. For each received block of length $N$, the corresponding input block is of length $N + \nu$ (due to the effect of channel memory). Hence, the channel matrix over this block is an $N \times (N + \nu)$ banded matrix denoted by $H^{(N)}$ whose rows are equal to shifted versions of $h^{(k)} : 1 \leq k \leq N$.

III Calculation of Achievable Rate

A Analysis of Mutual Information

In the presence of receiver side-information of the channel state, the mutual information over $K$ transmission blocks, each of size $N$, can be written as

$$I(X^K_1;Y^K_1;H^K_1) = \sum_{k=1}^{K} I(X_k;Y_k;H_k),$$  \hspace{1cm} (3)

where we have denoted the sequence of $K$ transmitted/received blocks by $X^K_1$, $Y^K_1$ and the channel realization is denoted by $H^K_1$. A coding theorem for a given transmission block size $N$ can be proved using standard arguments (see [2] for such an argument) and under assumptions of stationarity and ergodicity of the channel process $\{H_k\}$, the mutual information (and hence the achievable rate) can be written as

$$I_N = \lim_{K \rightarrow \infty} \frac{1}{N \, K} I(X^K_1;Y^K_1;H^K_1)$$  \hspace{1cm} (4)

$$= \frac{1}{N} \lim_{K \rightarrow \infty} \left[ \frac{1}{K} \sum_{k=1}^{K} I(X_k;Y_k;H_k) \right]$$  \hspace{1cm} (5)

$$= \frac{1}{N} \EE[I(X;Y;H)] \left( s \right) = \frac{1}{N} \EE[I(X;Y|H)],$$

where $(s)$ follows as the transmitter does not have channel side-information [2]. One question is whether $\lim_{N \rightarrow \infty} I_N$ exists and how does it behave with respect to amount of time-variation in the channel block $H^{(N)}$. This limit of large transmission block size is exactly where the block time-invariance assumption fails and therefore is the focus of our attention. Under the assumption of stationarity and ergodicity of the channel taps $\{h^{(k)}\}$, using techniques similar to [6] we can show that $\lim_{N \rightarrow \infty} I_N$ exists. In fact, using this argument, it can be shown that the random process $\frac{1}{N} I(X;Y|H)$ converges almost surely and therefore there is a convergence on a sample-path basis. However, we do not have an explicit form for the limit and therefore develop a recursion to calculate it numerically.

B Recursive Formula for Mutual Information

Using the fact that we have a white Gaussian input codebook and additive white Gaussian noise, we can write

$$I_N = \frac{1}{N} \EE \log \det (I_N + SNR \cdot H^{(N)} \cdot H^{(N)} *)$$

$$\overset{def}{=} \log SNR + \frac{1}{N} \EE \log \det R^{(N)},$$  \hspace{1cm} (6)

where $\det(.)$ denotes the determinant of a matrix. $I_N$ is the identity matrix of size $N$, $(.)^*$ denotes the complex-conjugate transpose operation, and $R^{(N)} \overset{def}{=} \frac{1}{N} \EE[I_N + H^{(N)} H^{(N)} *]$. It can readily checked that $H^{(N)}$ admits the partitioning

$$H^{(N)} = \begin{bmatrix} H^{(N-1)} & 0_{N \times (N-1)} \\ 0_{1 \times (N-1)} & h_1^{(N)} \end{bmatrix} \begin{bmatrix} h_1^{(N)} \\ h_2^{(N)} \end{bmatrix},$$  \hspace{1cm} (7)

where $0_{i \times j}$ is the all–zeros matrix with $i$ rows and $j$ columns. Therefore, the matrix $R^{(N)}$ defined in (6)
can be partitioned as follows
\[
\mathbf{R}^{(N)} = \begin{bmatrix}
\mathbf{R}^{(N-1)} & \mathbf{U}^{(N-1)} \\
\mathbf{U}^{(N-1)} & \mathbf{S}^{(N-1)}
\end{bmatrix}
\]
where \(|.|\) denotes the absolute value. Using the formula for the determinant of a 2 \times 2 block matrix, we have
\[
det(\mathbf{R}^{(N)}) = det(\mathbf{R}^{(N-1)}). \left\{ \left( \frac{1}{SNR} + (|h_1^{(N)}|^2 + |h_2^{(N)}|^2) \right) \right\}
- \left\{ (|h_1^{(N)}||h_2^{(N-1)}|)^2 (\mathbf{R}^{(N-1)})^{-1}(N-1, N-1) \right\}.
\]
where \(\mathbf{R}^{(N-1)} \overset{df}{=} \frac{1}{SNR} \mathbf{I}_N^{-1} + \mathbf{H}^{(N-1)} \mathbf{H}^{(N-1)^*} \). Using Kramer’s rule for matrix inversion, we have
\[
(\mathbf{R}^{(N-1)})^{-1}(N-1, N-1) = \frac{\text{Adj}(\mathbf{R}^{(N-1)})(N-1, N-1)}{\text{det}(\mathbf{R}^{(N-1)})} = \frac{\text{det}(\mathbf{R}^{(N-2)})}{\text{det}(\mathbf{R}^{(N-1)})}.
\]
Combining (6), (9), and (10), we have
\[
\mathcal{I}_N = \frac{N-1}{N} \mathcal{I}_N + \frac{1}{N} \log SNR
+ \frac{1}{N} \mathbb{E} \log \left( \frac{1}{SNR} + (|h_1^{(N)}|^2 + |h_2^{(N)}|^2) \right)
- \left( |h_1^{(N)}||h_2^{(N-1)}| \right)^2 \frac{\text{det}(\mathbf{R}^{(N-2)})}{\text{det}(\mathbf{R}^{(N-1)})}.
\]
Define the Cholesky factorizations
\[
\mathbf{R}^{(N)} \overset{df}{=} \mathbf{L}^{(N)} \mathbf{D}^{(N)} \mathbf{L}^{(N)^*} \quad \mathbf{R}^{(N-1)} \overset{df}{=} \mathbf{L}^{(N-1)} \mathbf{D}^{(N-1)} \mathbf{L}^{(N-1)^*},
\]
and note that due to the nesting property of Cholesky factorization, \(\mathbf{L}^{(N-1)}\) and \(\mathbf{D}^{(N-1)}\) are the leading sub-matrices of \(\mathbf{L}^{(N)}\) and \(\mathbf{D}^{(N)}\), respectively. Hence
\[
\frac{\text{det}(\mathbf{R}^{(N)})}{\text{det}(\mathbf{R}^{(N-1)})} = \prod_{i=1}^{N} d_i/d_{i-1} = d_N.
\]
\[
\text{det} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \text{det}(A)\text{det}(D - CA^{-1}B),
\]
\[
\text{det} \begin{bmatrix} A & \beta \end{bmatrix} \begin{bmatrix} \alpha \end{bmatrix} = \frac{\text{det}(A)}{\text{det}(\alpha)} \text{det}(A) = \text{det}(A) = \text{det}(A)\text{det}(D - CA^{-1}B).
\]
\[
\alpha_i = \frac{1}{SNR} + |h_i^{(i)}|^2,
\]
\[
\beta_i = |h_i^{(i)}|^2,
\]
\[
d_i = \alpha_i
\]
\[
\text{det}(\mathbf{L}^{(N)}) = 1. \quad \text{Using Equation (9), (10) and (12), we can develop the following computationally-efficient recursive formula for the } d_i
\]
\[
d_i = \alpha_i - \frac{\beta_i}{d_{i-1}} : \quad \text{for } i = 2, 3, \ldots, N
\]
\[
\text{where}
\]
\[
\alpha_i = \frac{1}{SNR} + |h_i^{(i)}|^2,
\]
\[
\beta_i = |h_i^{(i)}|^2,
\]
\[
d_i = \alpha_i
\]
\[
\text{C Alternative Mutual Information Expressions}
\]
It follows from (6), (11), and (13) that
\[
\mathcal{I}_N = \log SNR + \frac{1}{N} \mathbb{E} \sum_{i=1}^{N} \log(d_i)
\]
\[
= \log SNR + \frac{1}{N} \mathbb{E} \sum_{i=1}^{N} \log \left( \frac{1}{SNR} + |h_i^{(i)}|^2 \right)
- \frac{|h_1^{(i)} h_2^{(i-1)}|^2}{d_{i-1}}
\]
\[
= \log SNR + \frac{1}{N} \mathbb{E} \log \left( \frac{1}{SNR} + |h_i^{(i)}|^2 \right)
\]
\[
\text{Note that (15) has the same form as the mutual information of a flat-fading channel except that the channel vector energy at time } i \text{ is modified to}
\]
\[
|h_i^{(i)}|^2 = \left\{ \begin{array}{ll}
|h_i^{(i)}|^2 & \text{if } i = 1 \\
|h_i^{(i)}|^2 - \frac{|h_1^{(i)} h_2^{(i-1)}|^2}{d_{i-1}} & \text{if } i = 2, \ldots, N
\end{array} \right.
\]
By considering the following eigen-decomposition
\[
\mathbf{R}^{(N)} = \frac{1}{SNR} \mathbf{I}_N + \mathbf{H}^{(N)} \mathbf{H}^{(N)^*} = \mathbf{U}^{(N)} \mathbf{\Sigma}^{(N)} \mathbf{U}^{(N)^*},
\]
we arrive at the mutual information expression
\[
\mathcal{I}_N = \log SNR + \frac{1}{N} \mathbb{E} \sum_{i=1}^{N} \log(\sigma_i).
\]
As there is a sample-path convergence, to evaluate \(\lim_{N \to \infty} \mathcal{I}_N\), we examine the sample-path limit of the random process \(\frac{1}{N} \sum_{i=1}^{N} \log(\sigma_i)\). Note that it is significantly more complex to evaluate \(\frac{1}{N} \sum_{i=1}^{N} \log(\sigma_i)\) for large block size than \(\frac{1}{N} \sum_{i=1}^{N} \log(d_i)\) since eigenvalues do not have simple recursions as in (13).
IV Special Cases

A Flat-Fading Channel

For a flat-fading channel, the channel impulse response consists of a single tap which is assumed to have a variance equal to 1 (for a fair comparison with 2-tap channels). In this case, the mutual information is given by

$$I_N^{\text{flat}} = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \log(1 + SNR|h_i|^2).$$

(19)

At high SNR, we can evaluate the mutual information of flat-fading channels as follows

$$\lim_{N \to \infty} I_N^{\text{flat, high SNR}} = \log(SNR) + \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \log|h_i|^2.$$

The second term can be evaluated in closed form as done in [2].

B High-SNR Case

At high SNR, (6) simplifies to

$$I_N^{\text{high SNR}} \approx \log SNR + \frac{1}{N} \mathbb{E} \log \det(H(N)H^*(N)).$$

(20)

Since we are interested in the limit as $N$ becomes infinite, we can approximate $H(N)$ by an $N \times N$ matrix $H(N)$ constructed by wrapping around the last $\nu$ columns of $H(N)$.$^4$ For $\nu = 1$, the resulting $H(N)$ is an $N \times N$ bi-diagonal matrix whose determinant is given by

$$\det(H(N)) = \left( \prod_{i=1}^{N} h_1^{(i)} \right) + (-1)^{N+1} \left( \prod_{i=1}^{N} h_2^{(i)} \right).$$

(21)

Combining (20) and (21) we get

$$\lim_{N \to \infty} I_N^{\text{high SNR}} \approx \log SNR + \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \left| \det(H(N)) \right|^2$$

$$\approx \log SNR + \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \sum_{i=1}^{N} \log|h_1^{(i)}|^2$$

$$+ \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \left| 1 + (-1)^{N+1} \left( \prod_{i=1}^{N} h_2^{(i)} \right) \right|^2$$

$$= \lim_{N \to \infty} I_N^{\text{flat, high SNR}} - 1$$

$$+ \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \left| 1 + (-1)^{N+1} \left( \prod_{i=1}^{N} h_2^{(i)} / h_1^{(i)} \right) \right|^2.$$  

(22)

C Effect of Time Correlation

To model the effect of time correlation of the individual channel taps, we assume an auto-regressive (AR) model. More specifically, we assume that the time variation of the 2 channel taps follows the model

$$h_j^{(i+1)} = ah_j^{(i)} + z_j^{(i+1)} : \text{ for } j = 1, 2 \text{ and } i = 0, 1, \cdots$$

where $a$ is a correlation factor and $z_j = \begin{bmatrix} z_j^{(1)} \ z_j^{(2)} \cdots \end{bmatrix}$ are zero-mean Gaussian random processes with variance equal to $1 - a^2$. This makes the two channel vectors $h_j$ for $j = 1, 2$ zero-mean unit-variance AR Gaussian random processes with auto-correlation sequence given by $r_{nh}(k) = \mathbb{E} [h_j^{(i)} h_j^{*(i+k)}] = a^{|k|}$. To ensure stability (and hence stationarity and ergodicity) of the AR process, $0 \leq |a| < 1$. As $|a|$ approaches 1, the channel tap process becomes more correlated (i.e. slower fading) and vice versa.

D Effect of Channel Tap Correlation

In many practical scenarios, the channel taps (at any time instant) are correlated. This can occur, for example, due to the presence of a transmit filter. To isolate and investigate the effect of this correlation, we assume the following simple model. At time instant $i$, the 2 channel taps are modeled as 2 zero-mean unit-variance Gaussian random variables with cross-correlation coefficient $b$.

V Numerical Results

For our numerical results, all logarithms are computed base 2, i.e., $I_N$ is in bits/symbol.

In Figure 1 we demonstrate the sample path convergence of $\frac{1}{N} \left| I(X; Y | H) \right|$ asymptotic in the block length $N$.

Figure 2 illustrates the increase of the achievable rate of the 2-tap channel with the time correlation factor $a$. This can be attributed to the increase in the mean and decrease in the variance of $Elog(d_i)$ as $a$ increases, as illustrated in the histograms of Figure 3.

Introducing correlation between the two channel taps decreases the achievable rate, as shown in Figure 4.

Finally, Figure 5 shows that the mutual information of the 1-tap and 2-tap channels are generally different. This in contrast to the conclusions made in [2, 1] under the block-invariant channel model.

$^4$For time-invariant channels, this corresponds to approximating the Toeplitz linear convolution matrix by a circulant matrix [7].
VI Conclusions and Open Issues

A computationally- efficient recursive formula for computing the mutual information for 2-tap channels has been derived. Unlike the block-invariant channel model case [2, 1], the achievable rate of 1-tap and 2-tap channels may not be equal when the channels vary within the block. We found, through numerical calculations, that a higher tap time-correlation could increase the achievable rate of a 2-tap channel. Furthermore, increasing the correlation between the 2 time-varying channel taps could decrease the achievable rate. However, there is a significant caveat along with these numerical observations. The optimal input spectrum is not known for arbitrary time/tap correlations. Therefore, the white input may be highly suboptimal and may account for some of these effects. Therefore, we have not drawn strong conclusions from these numerical results. A challenging open problem is a closed-form characterization for the asymptotic eigenvalue distribution of random banded matrices which arise in frequency-selective channels modeled as FIR filters. Such a result would open the door for a more rigorous analytical framework for the ergodic capacity of time-varying frequency-selective channels. There are several open issues that might be of interest.

- The optimal covariance for maximizing mutual information for time-varying ISI channels. For the block time-invariant model this is known to be the Fourier basis with flat input spectrum (for independent taps) [2].
- An analytical characterization of the ergodic capacity of time-varying ISI channels.
- The precise characterization of the effect of time-correlation (or Doppler spread) on the ergodic capacity. This could answer one question of whether larger time-correlation (higher Doppler spread) increases the ergodic capacity.

References


Such asymptotic results are given in [8] only for full (i.e. non banded) random matrices.
Figure 3: Histograms of the $\log(d_i)$ for 100 Realizations of 2-Tap Channels with Low ($a = 0.1$) and High ($a = 0.9$) Tap Correlation Factors

Figure 4: Effect of Tap Correlation Factor on Mutual Information

Figure 5: Mutual Information of 2-Tap and 1-Tap Channels Versus Input SNR for Small and Large Time Correlation Factors


